Lab 5

Moments of Inertia

A. Purpose

To measure the moments of inertia of a disk and to determine the dimensions and material composition of a hidden object by its moments of inertia.

B. Introduction

The motion of a many-particle system is complicated, which involves the linear motion of the center of mass and the relative motion to the center of mass. The first step to understanding this complicated system is to discuss a simpler case, the rigid body. Specifically, we need to understand the physical laws of the rotational motion. This experiment focuses on the measurement of the moment of inertia, a measure of how easy or difficult it is to accelerate the rotational motion of a body, and further applies it to determine the properties of a hidden object.

For a rigid body (many-body system) rotating around a fixed axis, as shown in Fig. 1, the kinetic energy is

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \equiv \frac{1}{2} I \omega^2 \tag{1}$$

where ω is angular velocity and I is moment of inertia. This equation is similar to the expression of the linear kinetic energy

$$K = \frac{1}{2} M v^2$$

with the exchange $I \Leftrightarrow M$ and $\omega \Leftrightarrow v$.

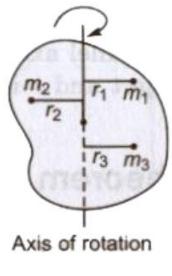


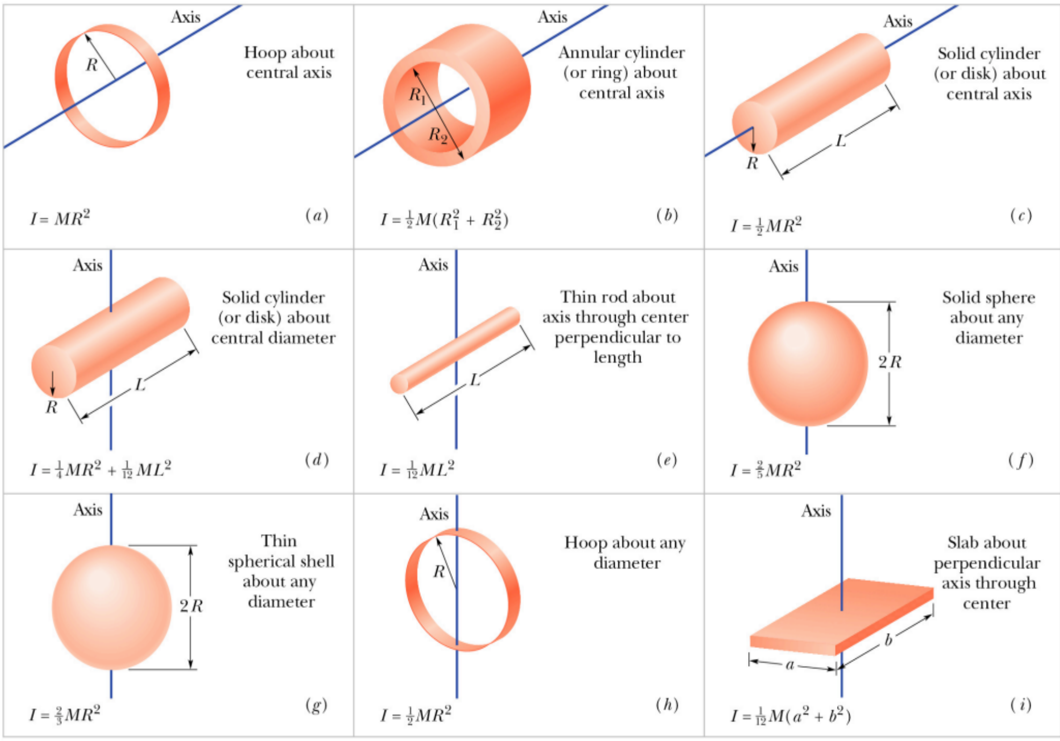
Fig. 1: Rotation of a rigid body

For a continuous body, the moment of inertia can be defined similarly as

$$I \equiv \int r^2 dm = \int r^2 \rho dV \tag{2}$$

Table 1 lists the moment of inertia of objects with different shapes with the chosen axes of rotation indicated by blue lines.

Table 1. Moments of inertia of objects with different shapes



If the rotational axis is now parallel to an axis through the center of mass, but displaced from it by a distance d, one can rewrite (3.1) as

$$K = K_{\mathrm{CM}} + K_{\mathrm{rel}} = \frac{1}{2}Mv_{\mathrm{CM}}^2 + \frac{1}{2}I_{\mathrm{CM}}\omega^2 = \frac{1}{2}\left(Md^2 + I_{\mathrm{CM}}\right)\omega^2 \equiv \frac{1}{2}I\omega^2 \tag{3}$$

where the kinetic energy is separated into two parts: the kinetic energy of the center of mass (CM) motion and the rotational kinetic energy around the CM. Note that when a rigid body rotates around a given axis, which is no need to be through the CM, the body itself would rotate at the same rate with respect to its CM. Therefore, the moment of inertia becomes

$$I = I_{\mathrm{CM}} + M \cdot d^2 \tag{4}$$

where M is the mass of the rigid body and d is the distance between the two axes. Eq. (4) is called the "Parallel Axis Theorem."

In general, the moments of inertia are calculated by integration; however, for some objects with symmetries, there is an alternative[1] to obtain the moment of inertia without integration. That is to divide the given object into several smaller (but not infinitesimal) objects, of which the moment of inertia is analogous to that of the complete object or is already known. While these pieces may have different shapes, here the simplest example, the moment of inertia of a thin rod around its center, is demonstrated, where every piece has the same shape as the original object.

Consider a thin, uniform rod of mass M and length L as Fig. 2 shows. From

[1] Benjamin Oostra, "Moment of Inertia Without Integrals", The Physics Teacher 44, 283-285 (2006) https://doi.org/10.1119/1.2195398

dimensional analysis, its moment of inertia around an axis passing through its center of mass and perpendicular to the length of the rod must take the form

$$I = \alpha ML^2 \tag{5}$$

where α is a dimensionless constant and the problem now is to find its value. For this purpose, the rod is divided into two rods of length $L/2$ and mass $M/2$. Note that any number of pieces will do, but two is enough and is also the simplest option.

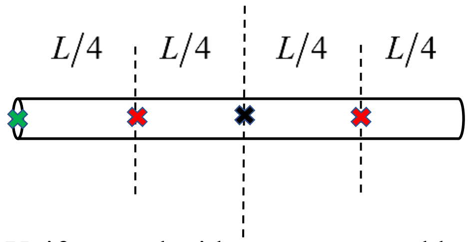


Fig. 2: Uniform rod with mass M and length L

Each of the two pieces has its center of mass at a distance $D = L/4$ from the original axis, so the moment of inertia I_1 of each piece (around the center of the complete rod) is

$$I_1 = \alpha(M/2)(L/2)^2 + (M/2)D^2 = \alpha(M/2)(L/2)^2 + (M/2)(L/4)^2 \tag{6}$$

Now, impose the condition that $I = 2I_1$, then the value of α can be obtained by

$$\alpha ML^2 = 2\left\{\alpha(M/2)(L/2)^2 + (M/2)(L/4)^2\right\} \tag{7}$$

Therefore, $\alpha = 1/12$.

This idea can be applied to the moment of inertia of a thin rod around one of its ends. Again, the result must take the form like eq (5), and the problem is to find α. This time, the moment of inertia I_2 of the left piece (around its left end, which is also the left end of the entire rod) is

$$I_2 = \alpha(M/2)(L/2)^2 \tag{8}$$

The moment of inertia of the right half, around its left end, is the same, but to compute its moment around the original axis, recall the parallel-axis theorem. The right half part of the rod is like its left part shifted by a distance $L/4$ to the left and then $3L/4$ to the right. Therefore,

$$I_3 = \alpha(M/2)(L/2)^2 - (M/2)(L/4)^2 + (M/2)(3L/4)^2 \tag{9}$$

Finally, impose the condition that $I = I_2 + I_3$, then $\alpha = 1/3$ is obtained, as expected.

To obtain the moments of inertia experimentally, as illustrated in Fig. 3, a known torque is applied to the rotary body so that with the measurement of the responding angular acceleration, the moment of inertia can be obtained. Moreover, the known torque can be created by hanging a mass from a string wound onto the step pulley. When the hanging mass is released, it falls down, making the step pulley rotate around the axis due to the tension of the string. Therefore,

$$\vec{\tau} = \vec{r} \times \vec{T} \tag{10}$$

where r is the radius of the step pulley and T is the tension. With Newton's Second Law and the relation between the Linear and Rotational acceleration, the tension is obtained,

$$mg - T = ma \Rightarrow T = m(g-a) = m(g-r\alpha) \tag{11}$$

Therefore, the moment of inertia is solved by

$$\tau = rT = mr(g-r\alpha) = I\alpha \Rightarrow I = mr\left(\frac{g}{\alpha} - r\right) \tag{12}$$

where α is the angular acceleration of the object.

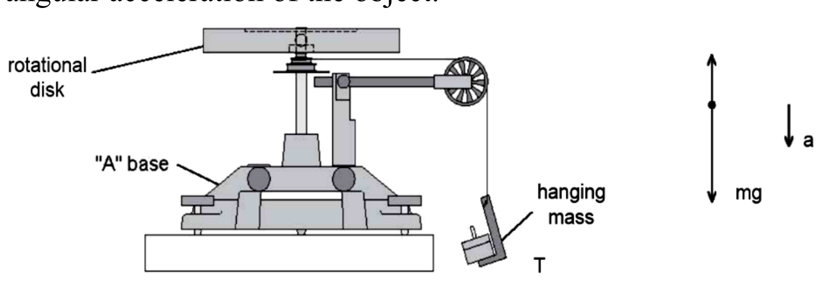


Fig. 3: Experimental setup

Another method to obtain the moments of inertia is to use the "torsion pendulum." The working principle of the torsion pendulum is left as a pre-lab assignment.

C. Apparatus and software

1. Software to use:
   (1) Drive for Arduino Mega Case: https://www.arduino.cc/en/software
   (2) CoolTerm: http://freeware.the-meiers.org
2. Experimental setup

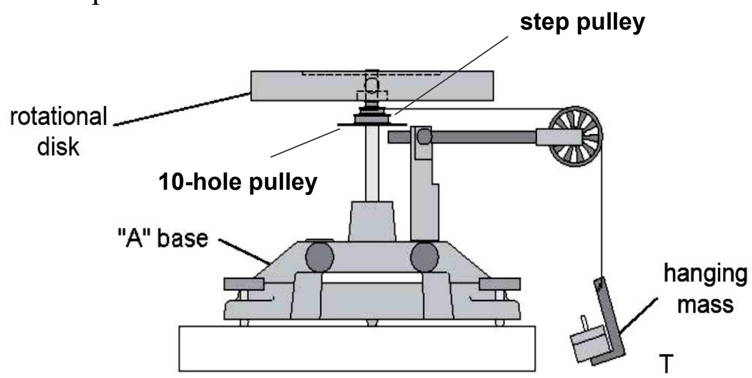


Fig. 4: Moment of inertia of the horizontal disk

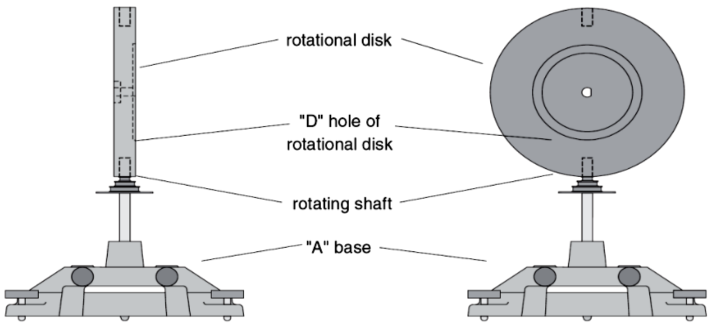


Fig. 5: Moment of inertia of the vertical disk

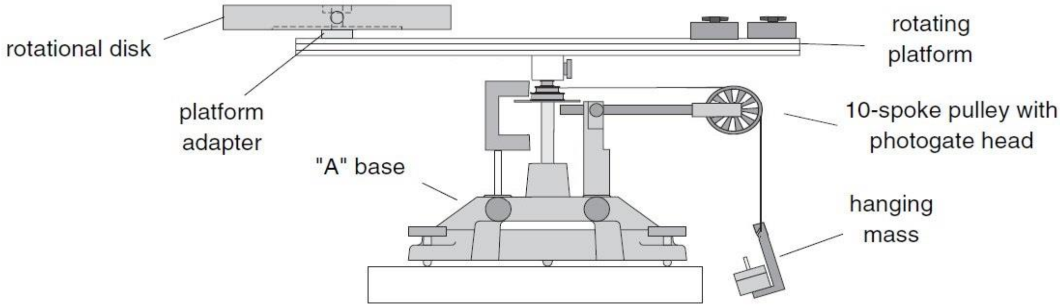


Fig. 6: Moment of inertia of the off-axis disk

D. Procedures

1. Pre-lab assignments (hand in before the lab)

 (1) Download the software needed for this experiment in advance

 (2) Make a flowchart of this experiment and answer the questions

  (i) Use the idea introduced in the theory section to determine the moment of inertia of an equilateral triangle around a perpendicular axis passing through its center with mass M and height H. That is, you should find the value of α for this time. (Hint: Subdivide the triangle into four smaller equilateral triangles, each of mass $M/4$ and height $H/2$.)

  (ii) As described in the theory section, to obtain the moment of inertia of an object, its angular acceleration α under a fixed torque, is measured. Specifically, the Arduino Mega Case is used to store the data obtained by the photogate. Suppose you are asked to conduct an experiment of obtaining the moment of inertia of a Horizontal Disk around its center. The experimental setup is as Fig. 4 shows, and the data collected is shown in the table below.

   (a) If the rotational friction occurs and causes an equivalent negative angular acceleration α_F during the experiment, find the moment of inertia of the system by considering α_F. That is, modify eq. (12) with α_F.

   (b) You can see from the table below that the data obtained through the photogate are T_1 and T_2, which represent the time the infrared beam gets through the 10-hole pulley and the time it gets interrupted by the 10-hole pulley, respectively, as shown by Fig. 6. Also, the average frequency f and the average period T of the rotation within the time range from T_1 to T_2, are calculated and given in the table. Draw a graph of f versus $(T_1+T_2)/2$, and explain how you can obtain the angular acceleration by the slope of the best fitting line. Also, the mass of hanging mass and the diameter of the step pulley wound by the string are $m=198.61$ g and $D=1.645$ cm. (Hint: $\omega=2\pi f$)

T_1 (ms)	T_2 (ms)	f $(1/s)$	T (s)
589	685	1.042	0.96
870	959	1.124	0.89
1131	1215	1.190	0.84
1379	1459	1.250	0.80
1614	1689	1.333	0.75
1837	1909	1.389	0.72
2051	2120	1.449	0.69
2256	2324	1.471	0.68
2454	2519	1.538	0.65
2646	2708	1.613	0.62
2831	2892	1.639	0.61
3011	3069	1.724	0.58
3185	3243	1.724	0.58
3356	3412	1.786	0.56
3522	3576	1.852	0.54
3684	3737	1.887	0.53
3843	3895	1.923	0.52
3997	4048	1.961	0.51
4149	4198	2.041	0.49
4297	4345	2.083	0.48

(c) Suppose the equivalent negative angular acceleration is given by $\alpha_F = -0.0452\ rad/s^2$. Find the moment of inertia of the horizontal disk by considering the friction, and compare the result with the theory. Note the mass of the disk is $M_{disk} = 1556.30$ g and its diameter is $D_{disk} = 22.8$ cm.

(Question: How to obtain α_F experimentally? Hint: Same idea as a cart sliding on the track in Lab 2.)

(iii) For a torsion pendulum, its torsion wire is essentially inextensible but free to twist about its axis. Any twisting of the wire is associated with mechanical deformation, which the wire would resist by developing a restoring torque τ, acting to restore the wire to its untwisted state. For relatively small angles of twist, the magnitude of this torque is directly proportional to the twist angle,

$$\tau = -\kappa\theta$$

where $\kappa > 0$ is the torque constant of the wire. Compare the above equation to Hooke's law, and find the period in terms of κ and I, the moment of inertia of the object hung under the torsion pendulum. Also, draw a schematic for the torsion pendulum. (Do not forget to give the reference, if you copy one online.)

2. In-lab activities

(1) Set up the equipment.

(i) Level the Base

(a) Purposely make the apparatus unbalanced by attaching the 300 g square mass onto either end of the track and tighten the screw so the mass won't slide. Note that if the hooked mass is hanging from the side post in the centripetal force accessory, place the square mass on the same side.

(b) Adjust the leveling screw on one of the legs of the base until the end of the track with the square mass is aligned over the leveling screw on the other leg of the base. See Fig. 7.

(c) Rotate the track 90 degrees so it is parallel to one side of the "A" and adjust the other leveling screw until the track stays in this position

(d) The track is now level, and it should remain at rest in any orientation

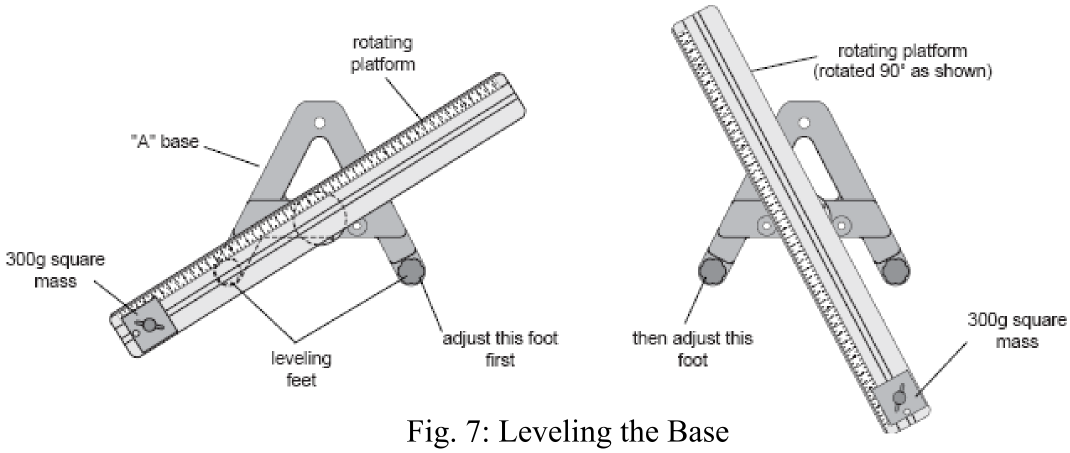


Fig. 7: Leveling the Base

(ii) Mount the Photogate, as shown in Fig. 8 (a).

(a) Slide the non-threaded end of the photogate mount rod into a hole in the A-base and clamp it in place with the thumbscrew

(b) Adjust the Photogate so that its infrared beam can be interrupted by the 10-hole pulley on the vertical shaft as the shaft turns

(c) Connect the cable to the Photogate and Arduino Mega Case, and when the Photogate is powered, you can tell when the photogate is blocked by watching the LED indicator on the photogate

(iii) Install the Super Pulley, as shown in Fig. 8 (b).

(a) Insert the Super Pulley Rod into the hole in the black rod and tighten the set screw against the Super Pulley Rod.

(b) Adjust the position of the base so the string passing over the Super Pulley will clear the edge of the table

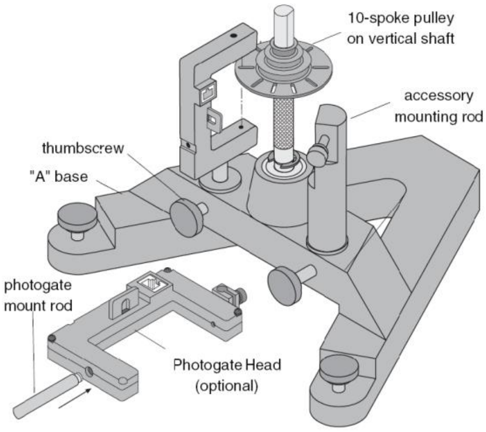


Fig. 8(a)

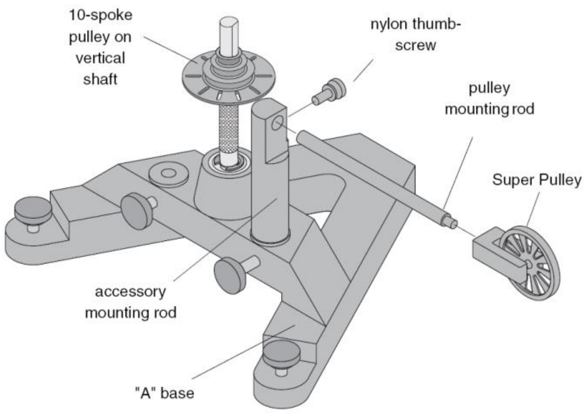


Fig. 8(b)

(iv) Assemble moment of inertia accessory

The rotational disk can be placed directly onto the axle of the rotating base or can be used with the rotating platform via the included platform adapter.

(a) Attach the square nut (supplied with the moment of inertia accessory) to the platform adapter.

(b) Position the platform adapter at the desired radius as shown in Fig. 9.

(c) Grip the knurled edge of the platform adapter and tighten.

The rotating disk can be mounted in a variety of positions using any of the four holes on the rotation disk.

(a) Two "D" holes exist on the edge of the disk, located at 180° from one another.

(b) One "D" hole is located at the center on the top surface (the surface with the metal ring channel and the label) of the disk.

(c) One hole is located at the center on the bottom surface of the disk and is actually the inner race of a bearing. This enables the rotational disk to rotate (in either direction) in addition to other rotating motions applied to your experiment setup.

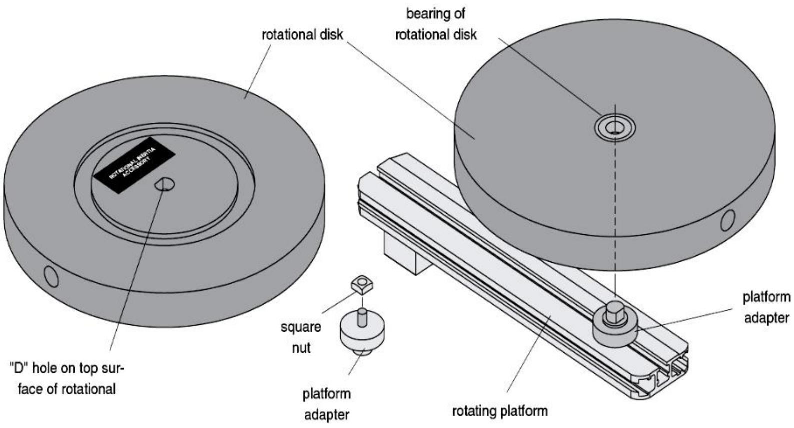


Fig. 9: Moment of inertia accessory including platform adapter assembly

(2) Moment of inertia of the horizontal disk

Remove the track from the Rotating Platform and directly place the disk on the center shaft as shown in Fig. 4. Connect the Photogate/ Pulley System to the Arduino Box that is powered by the computer. Find a way to measure the equivalent negative rotational acceleration caused by the friction. Use anything you need to obtain the moment of inertia of the horizontal disk both experimentally and theoretically, and find the percentage difference between them.

(3) Moment of inertia of the vertical disk

Change the horizontal disk to the vertical disk, and put it on the center shaft as shown in Fig. 5. Connect the Photogate/ Pulley System to the Arduino Box that is powered by the computer. Find a way to measure the equivalent negative rotational acceleration caused by the friction. Use anything you need to obtain the moment of inertia of the vertical disk both experimentally and theoretically, and find the percentage difference between them.

(4) Moment of inertia of the off-axis disk

Set up the Rotational Accessory as shown in Fig. 6. Mount the disk with its bearing side up. Use the platform adapter to fasten the disk to the track at a large radius. Connect the Photogate/ Pulley System to the Arduino Box that is powered by the computer. Find a way to measure the equivalent negative rotational acceleration caused by the friction. Use anything you need to obtain its moment of inertia of the off-axis disk both experimentally and theoretically, and find the percentage difference between them.

***Note: Five independent measurements are needed for this part. Draw a graph of I_{off} versus d^2 and use the best fitting line to verify the parallel-axis theorem, where $I_{\text{off}} \equiv I_{\text{ex}} - I_{\text{disk}}$, defined as the extra moment of inertia resulting from off-axis displacement and d is the distance between the center axis of rotation and the disk.

(5) Black-box measurement by the torsion pendulum

Consider the setup in Fig. 10. Suppose you have a calibration object with a known moment of inertia I_{cal} calculated from its mass and spatial dimensions. Find a way to measure the moment of inertia of the given hidden object and to tell its shape.

***Note: Measure the elapsed time for 50 oscillations of the calibration object as well as the black box, and use the equation mentioned in the pre-lab to determine.

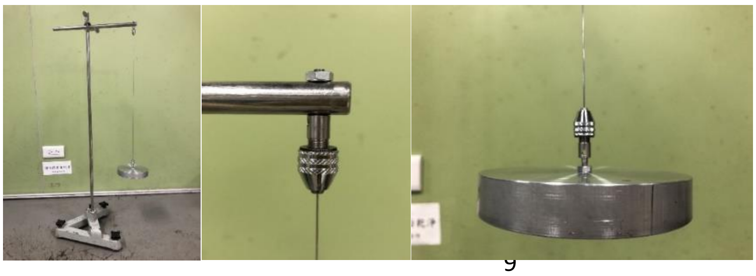


Fig. 10: Experimental setup of torsion pendulum with a hanging calibration object.

3. Post-lab report
   (1) Recopy and organize your data from the in-lab tables in a neat and more readable form.
   (2) Analyze the data you obtained in the lab and answer the given questions
   (3) Compare the results with the theory, and discuss the uncertainties that occur in the experiments, and how they influence the experiments. (Quantitatively, if possible.)

E. Questions

1. In the theory section, the Parallel-Axis Theorem is introduced. There is another theorem called Perpendicular-Axis Theorem, which states the relation among the moments of inertia of a thin object around three orthogonal axes. Google and explain it in words with an example. (Note: Rewrite the idea in your own words.)
2. While calculating the theoretical moments of inertia for the disk, we didn’t consider it thickness. Estimate the percent difference caused by this assumption.

F. References

Chapter 6: Rotational kinetics, Kuo-An Wu, General Physics(I), NTHU OCW (2017). https://reurl.cc/gWqRn7
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Aghamohammadi, Amir. "Moment of inertia: tricky ways." European Journal of Physics 43.3 (2022): 035005.
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